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# MINIMUM GRAPHS WITH COMPLETE $k$-CLOSURE

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The $k$-closure of a graph $G$, as defined by Bondy and Chvátal, is the graph obtained from $G$ by recursively joining pairs of nonadjacent vertices with degree-sum at least $k$. We determine the number of edges necessary for the $k$-closure of a graph to be complete.

## 1. The theorem

We follow the notation and terminology of Bondy [2] except that we do not allow multiple edges or loops and we use $n$ to denote the number of vertices of a graph.

Given a graph $G$ of order $n$ and a nonnegative integer $k$, there is a unique smallest supergraph $H$ of order $n$ such that $d_H(u)+d_H(v)<k$ for all $uv\notin E(H)$. Bondy and Chvátal [1] have called this graph $H$ the *$k$-closure* of $G$ and denoted it by $C_k(G)$.

They further define a property $P$ to be *$k$-stable* if whenever $G+uv$ has property $P$ and $d_G(u)+d_G(v)\geqslant k$, then $G$ itself has property $P$. The stability of each of several properties is determined in [1]; in particular it is shown (by paraphrasing Ore's proof [5]) that the property of containing a hamiltonian cycle is $n$-stable.

Since $C_k(G)$ can be obtained from $G$ by recursively joining nonadjacent vertices with degree-sum at least $k$, they conclude that if $P$ is $k$-stable and $C_k(G)$ has property $P$, then $G$ itself has property $P$.

All of the properties, save one, whose stabilities were determined by Bondy and Chvátal are enjoyed by all appropriately large complete graphs; consequently it is of interest to know when it is possible for $C_k(G)$ to be complete. Our object is the determination of the minimum number of edges a graph $G$ of order $n$ can have if $C_k(G)$ is to be complete. We denote this number by $f(k)$ (it is, as we will argue, independent of $n$) and will show the following.

**Theorem.** $f(k) = \lfloor \frac{1}{8}(k+2)^2 \rfloor$ *for incomplete graphs.*

Before proving this result we note certain consequences of it.

**Remark 1.** In [1], Bondy and Chvátal described an algorithm for obtaining $C_k(G)$ from $G$ in $O(n^4)$ steps. If the degree-sum of the vertices of $G$ is less than $2\lfloor \frac{1}{8}(k+2)^2 \rfloor$ then, of course, there is no point in applying the algorithm with the hope of obtaining $C_k(G) = K_n$.

**Remark 2.** The Bondy–Chvátal citerion for the existence of a hamiltonian cycle is simply the following: $G$ is hamiltonian if any only if $C_n(G)$ is hamiltonian. Consequently for each $s$, $\lfloor \frac{1}{8}(n+2)^2 \rfloor \leqslant s \leqslant \binom{n}{2}$, there is a hamiltonian graph with $n$ vertices and $s$ edges satisfying the Bondy–Chvátal criterion. In marked contrast to this, a graph with fewer than $\frac{1}{4}n^2$ edges obviously cannot satisfy Dirac's criterion [4].

If a graph $G$ satisfies Ore's criterion [5] and $v$ is a vertex of $G$ with minimum degree $\delta$, then each of the $n-\delta-1$ vertices not adjacent to $v$ has degree at least $n-\delta$ so tnat $G$ has at least

$$\binom{n-\delta}{2} + \binom{\delta+1}{2} \geqslant \tfrac{1}{4}(n^2-1) \quad \text{edges.}$$

Suppose $G$ is a graph with degree sequence $d_1, \ldots, d_n$ such that $d_1 \leqslant \cdots \leqslant d_n$ and that Pósa's criterion [6] holds. Then $d_m > m$ for $m < \frac{1}{2}(n-1)$ and, if $n$ is odd, $d_{n-2} > \frac{1}{2}(n-1)$. Consequently $G$ has at least $\frac{1}{16}(3n^2+8n-11)$ edges if $n$ is odd and at least $\frac{1}{16}(3n^2+6n-8)$ edges if $n$ is even.

Let $1 < d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n \leqslant n-1$ be the degree sequence of a graph $G$ satisfying the criterion of Chvátal [3], i.e., $d_i \leqslant i < \frac{1}{2}n$ implies $d_{n-i} \geqslant n-i$. Then the interval of integers $[1, \lfloor \frac{1}{2}(n-1) \rfloor]$ has a unique partition into consecutive maximal subintervals $I_1, \ldots, I_k$ such that $1 \in I_1$ and for $1 \leqslant j \leqslant k$ either (i) $d_i \leqslant i$ for all $i$ in $I_j$ or (ii) $d_i > i$ for all $i$ in $I_j$. For each subinterval $I_j$ let $a_j$ and $b_j$ be the smallest and largest members of $I_j$, respectively. We note that $a_{j+1} = b_j + 1$ for $1 \leqslant j \leqslant k-1$ and define $a_{k+1} = b_k + 1 (= \lfloor \frac{1}{2}(n+1) \rfloor)$.

If $I$ is a subinterval satisfying (i), then $a-1 < d_{a-1} \leqslant d_i \leqslant i$ for $a \leqslant i \leqslant b$ so that $d_{n-i} \geqslant n-i$ and we have

$$\sum_{i=a}^{b} d_i + \sum_{i=a}^{b} d_{n-i} \geqslant \sum_{i=a}^{b} (a+n-i).$$

A similar argument shows that the same bound holds for subintervals satisfying (ii). Consequently

$$\sum_{i=1}^{n} d_i \geqslant \sum_{j=1}^{k} \sum_{i=a_j}^{b_j} (n+a_j-i) \geqslant n \lfloor \tfrac{1}{2}(n-1) \rfloor - \sum_{i=1}^{\lfloor (n-1)/2 \rfloor} (i-1)$$

so that $G$ has at least $\frac{1}{16}(3n^2-2n-8)$ edges.

The several bounds obtained above suggest that if the effectiveness of a criterion for the existence of a hamiltonian cycle in a graph $G$ is measured only by the number of edges needed for satisfaction of the criterion, then the improvement of the Bondy–Chvátal criterion over that of Pósa or Chvátal type is as significant as the improvement of the latter over that of Dirac or Ore type.

**Remark 3.** In applying the Bondy–Chvátal criterion to a graph $G$ we have, on occasion, found it advantageous to work with the complement of $G$. In the next section we emphasize, as propositions, some of the conspicuous properties of the complementary form of the Bondy-Chvátal criterion.

**Remark 4.** In the next section we will construct the complements of graphs $G$ of order $n$ with $C_k(G)=K_n$ and having $f(k)$ edges. There are more such extremal graphs but we have no estimates for the number. The determination of nontrivial bounds for this number appears to us to be an interesting problem of some significance.

## 2. Proof of the theorem

We find it easier to consider the complementary problem. For this purpose we say a graph $G$ is *m-vanishing*, $m \geq 0$, if the recursive deletion of edges $uv$ for which $d(u)+d(v) \leq m$ leads to a totally disconnected graph. For $m \geq 0$, $n \geq 1$, let $g(n, m)$ be the maximum number of edges possible in an $m$-vanishing graph of order $n$. We note that

$$g(n, m)=\binom{n}{2} \quad \text{if } m \geq 2n-2. \tag{1}$$

**Proposition 1.** *If $G$ is a nontrivial $m$-vanishing graph, then $G-v$ is $m$-vanishing for each vertex $v$ of $G$.*

This follows since the edges of $G-v$ may be deleted in the same order as that in obtaining a totally disconnected graph from $G$.

From Proposition 1 it follows that if $U$ is any proper subset of vertices of an $m$-vanishing graph then $G-U$ is also $m$-vanishing. In particular an $m$-vanishing graph $G$ of order $n \geq 3$ with at least one edge contains an edge $uv$ such that $d_G(u)+d_G(v) \leq m$ and $G-\{u, v\}$ is $m-$vanishing. Consequently we have

$$g(n+2, m) \leq m-1+g(n, m) \quad \text{for } m,n \geq 1. \tag{2}$$

**Proposition 2.** *If* $1 \leq m \leq 2n+1$, *then*

$$g(n, m) \leq \begin{cases} \frac{1}{2}n(m-1)-\frac{1}{8}(m^2-1), & m \text{ odd}, \\ \frac{1}{2}n(m-1)-\frac{1}{8}m^2, & m \text{ even}, n-\frac{1}{2}m \text{ even}, \\ \frac{1}{2}n(m-1)-\frac{1}{8}(m^2-4), & m \text{ even}, n-\frac{1}{2}m \text{ odd}. \end{cases} \tag{3}$$

**Proof.** We prove (3) by induction on $n$. If $m=2n-2$, $2n-1$, $2n$ or $2n+1$ it is easy to verify that the r.h.s. of (3) is equal to $\binom{n}{2}$ which, by (1), is $g(n, m)$. The inductive step follows immediately by using (2) so we conclude that (3) holds.

We will construct $m$-vanishing graphs of order $n$ that have the number of edges specified in the r.h.s. of (3) so that equality in (3) will be established.

Before doing so, however, let us suppose equality holds in (3) and determine $f(n, k)$, the minimum number of edges a graph of order $n$ with complete $k$-closure can have.

Now, for $0 \leq k \leq 2n-2$, a graph $G$ of order $n$ has a complete $k$-closure if and only if the complement of $G$ is $(2n-k-2)$-vanishing since the sequence of edges added to $G$ to obtain $K_n$ is precisely the sequence of edges deleted from $G^c$ to obtain $K_n^c$. As a consequence we have the following.

**Proposition 3.** *$f(n, k)=\binom{n}{2}-g(n, 2n-k-2)$ for $0 \leq k \leq 2n-2$.*

If we assume equality in (3) and set $m=2n-k-2$ we obtain, for $0 \leq k \leq 2n-2$,

$$f(n, k)=\begin{cases} \frac{1}{8}(k^2+4k+3), & k \text{ odd} \\ \frac{1}{8}(k^2+4k+4), & k \text{ even}, \tfrac{1}{2}k \text{ odd} \\ \frac{1}{8}(k^2+4k), & k \text{ even}, \tfrac{1}{2}k \text{ even}. \end{cases} \tag{4}$$

Since the theorem is an immediate consequence of (4) it remains only to construct $m$-vanishing graphs of order $n$, $1 \leq m \leq 2n+1$, having the number of edges specified in the r.h.s. of (3). We construct such graphs, $H(n, m)$, as follows.

For $n \geq 1$ let $H(n, 1)$ be $K_n^c$. For $n \geq 1$ let $H(n, 2)$ consist of $\lfloor \frac{1}{2}n \rfloor$ copies of $K_2$ and, if $n$ is odd, an additional vertex. For $n \geq 1$ let $H(n, 3)$ be the path on $n$ vertices. It is easily verified that $H(n, m)$, $m=1, 2, 3$, is $m$-vanishing and has the number of edges specified in the r.h.s. of (3). For $4 \leq m \leq 2n+1$ our construction depends on the parity of $m$.

(i) $m$ is even and at least 4. For both $l=\frac{1}{2}m$ and $l=\frac{1}{2}m+1$ set $H(l, m)=K_l$, and label its vertices $x_j$, $0 \leq j \leq l-1$. For $n=2i+l$, $i \geq 1$, we define $H(n, m)$ recursively by

$$V(H(n, m))=V(H(n-2, m)) \cup \{x_{-i}, x_{l+i-1}\}$$

and

$$\begin{aligned} E(H(n, m))=E(H(n-2, m)) &\cup \{x_{-i}x_{-i+1}, x_{-i}x_{l+i-1}, x_{l+i-1}x_{l+i-2}\} \\ &\cup \{x_{-i}x_j \mid l+i-\tfrac{1}{2}m \leq j \leq l+i-3\} \\ &\cup \{x_{l+i-1}x_j \mid -i+1 \leq j \leq \tfrac{1}{2}m-i-2\} \end{aligned} \tag{5}$$

and note that two of the sets are empty for $m=4$. It is easily verified that $m$-1 edges are added to $H(n-2, m)$ in constructing $H(n, m)$ so that the size of $H(n, m)$ is given by the r.h.s. of (3).

It remains, in this case, to show that $H(n, m)$ is $m$-vanishing. For this purpose we exhibit the degrees of the vertices of $H(n, m)$ in the following tables. That these actually are the degrees of the vertices of $H(n, m)$ may be proven by induction on $i$ using (5). Because of the many cases that must be considered we suppress the proof; full details will appear in the dissertation of the first author.

To use the tables to find $d(x_j)$ in $H(n, m)$ one first chooses $l$ to be $\frac{1}{2}m$ or $\frac{1}{2}m+1$ if $m$ is even and $\frac{1}{2}(m-1)$ or $\frac{1}{2}(m+1)$ if $m$ is odd so that $n$ and $l$ have the same parity. The values of $m$ and $l$ determine which table is to be used. Let $i=\frac{1}{2}(n-l)$ and, in the appropriate table, choose the pair of columns corresponding to the interval containing $i$. In the left one of these two columns find the interval containing (or value of) $j$ and to the right of it read $d(x_j)$.

For example to determine the degree of $x_{15}$ in $H(40, 10)$ we use the table for $m=10$ and $l=\frac{1}{2}m+1=6$, i.e., Table 2. We determine $i=17$ and so, use the third pair of columns. Since $j$ lies in the interval $[\frac{1}{2}m, i+1]$ we have $d(x_{15})=m-1=9$.

The table for $d(x_j)$ in $H(n, m)$, $n=2i+l$, $l=\frac{1}{2}m$, $m\equiv 0 \bmod 4$ has column-pair headings $1\leqslant i\leqslant\frac{1}{4}(m-4)$, $\frac{1}{4}m\leqslant i\leqslant\frac{1}{2}(m-6)$, $i=\frac{1}{2}(m-4)$ and $i\geqslant\frac{1}{2}(m-2)$; otherwise the table is the same as Table 1.

The table for $d(x_j)$ in $H(m, n)$, $n=2i+l$, $l=\frac{1}{2}m+1$, $m\equiv 0 \bmod 4$ is the same as Table 2 except that the column-pair headings read $1\leqslant i\leqslant\frac{1}{4}(m-8)$, $\frac{1}{4}(m-4)\leqslant i\leqslant\frac{1}{2}(m-6)$ and $i\geqslant\frac{1}{2}(m-4)$.

If from $H(n, m)$, $n>l$, we consecutively delete the edges $x_{-i}x_{l+i-1}$, $x_{l+i-1}x_{l+i-2}$, $x_{l+i-1}x_j$ for $-i+1\leqslant j\leqslant\frac{1}{2}m-i-2$, $x_{-i}x_{-i+1}$ and $x_ix_j$ for $l+i-3\geqslant j\geqslant l-\frac{1}{2}m+i$ we are left with $H(n-2, m)+x_{-i}+x_{l+i-1}$. From the tables we verify that we have deleted only edges $uv$ where $d(u)+d(v)\leqslant m$, the degrees being measured in the graph from which $uv$ is deleted. Consequently, in the case $m$ is even, we obtain an inductive proof that $H(n, m)$ is vanishing; as before, details of the proof are left to the dissertation of the first author.

(ii) $m$ is odd and at least 5. For both $l=\frac{1}{2}(m-1)$ and $l=\frac{1}{2}(m+1)$ set $H(l, m)=K_l$ and label its vertices $x_j$, $0\leqslant j\leqslant l-1$. For $n=2i+l$, $i\geqslant 1$, we define $H(n, m)$ recursively by

$$V(H(n, m))=V(H(n-2, m))\cup\{x_{-i}, x_{l+i-1}\}$$

Table 1. $d(x_j)$ in $H(n, m)$, $n=2i+l$, $l=\frac{1}{2}m$, $m\equiv 2 \bmod 4$.

| $1\leqslant i\leqslant\frac{1}{4}(m-6)$ | | $\frac{1}{4}(m-2)\leqslant i\leqslant\frac{1}{2}(m-6)$ | | $i=\frac{1}{2}(m-4)$ | | $i\geqslant\frac{1}{2}(m-2)$ | |
|---|---|---|---|---|---|---|---|
| $j$ | $d(x_j)$ | $j$ | $d(x_j)$ | $j$ | $d(x_j)$ | $j$ | $d(x_j)$ |
| $-i$ | $\frac{1}{2}m$ | $-i$ | $\frac{1}{2}m$ | $-i$ | $\frac{1}{2}m$ | $-i$ | $\frac{1}{2}m$ |
| $[-i+1, -1]$ | $\frac{1}{2}m+i+j+1$ | $[-i+1, -1]$ | $\frac{1}{2}m+i+j+1$ | $[-i+1, -1]$ | $\frac{1}{2}m+i+j+1$ | $[-i+1, \frac{1}{2}m-i-3]$ | $\frac{1}{2}m+i+j+1$ |
| $0$ | $\frac{1}{2}m+i$ | $0$ | $\frac{1}{2}m+i$ | $0$ | $m-2$ | $[\frac{1}{2}m-i-2, -1]$ | $m-1$ |
| $[1, i-1]$ | $\frac{1}{2}m+i+j-1$ | $[1, \frac{1}{2}m-i-3]$ | $\frac{1}{2}m+i+j-1$ | $[1, \frac{1}{2}m-1]$ | $m-3$ | $0$ | $m-2$ |
| $[i, \frac{1}{2}m-i-2]$ | $\frac{1}{2}m+2i-1$ | $[\frac{1}{2}m-i-2, i]$ | $m-3$ | $[\frac{1}{2}m, \frac{1}{2}m+i-2]$ | $m+i-j-1$ | $[1, \frac{1}{2}m-2]$ | $m-3$ |
| $[\frac{1}{2}m-i-1, \frac{1}{2}m-2]$ | $m+i-j-3$ | $[i+1, \frac{1}{2}m-2]$ | $m+i-j-3$ | $\frac{1}{2}m+i-1$ | $\frac{1}{2}m$ | $\frac{1}{2}m-1$ | $m-2$ |
| $\frac{1}{2}m-1$ | $\frac{1}{2}m+i-1$ | $\frac{1}{2}m-1$ | $\frac{1}{2}m+i-1$ | | | $[\frac{1}{2}m, 1]$ | $m-1$ |
| $[\frac{1}{2}m, \frac{1}{2}m+i-2]$ | $m+i-j-1$ | $[\frac{1}{2}m, \frac{1}{2}m+i-2]$ | $m+i-j-1$ | | | $[i+1, \frac{1}{2}m+i-2]$ | $m+i-j-1$ |
| $\frac{1}{2}m+i-1$ | $\frac{1}{2}m$ | $\frac{1}{2}m+i-1$ | $\frac{1}{2}m$ | | | $\frac{1}{2}m+i-1$ | $\frac{1}{2}m$ |

Table 2. $d(x_j)$ in $H(n, m)$, $n = 2i + l$, $l = \frac{1}{2}m + 1$, $m \equiv 2 \bmod 4$.

| $1 \leq i \leq \frac{1}{4}(m-6)$ | | $\frac{1}{4}(m-2) \leq i \leq \frac{1}{2}(m-6)$ | | $i \geq \frac{1}{2}(m-4)$ | |
|---|---|---|---|---|---|
| $j$ | $d(x_j)$ | $j$ | $d(x_j)$ | $j$ | $d(x_j)$ |
| $-i$ | $\frac{1}{2}m$ | $-i$ | $\frac{1}{2}m$ | $-i$ | $\frac{1}{2}m$ |
| $[-i+1, 0]$ | $\frac{1}{2}m+i+j+1$ | $[-i+1, 0]$ | $\frac{1}{2}m+i+j+1$ | $[-i+1, \frac{1}{2}m-i-3]$ | $\frac{1}{2}m+i+j+1$ |
| $[1, i]$ | $\frac{1}{2}m+i+j-1$ | $[1, \frac{1}{2}m-i-3]$ | $\frac{1}{2}m+i+j-1$ | $[\frac{1}{2}m-i-2, 0]$ | $m-1$ |
| $[i+1, \frac{1}{2}m-i-2]$ | $m+2i$ | $[\frac{1}{2}m-i-2, i+1]$ | $m-3$ | $[1, \frac{1}{2}m-2]$ | $m-3$ |
| $[\frac{1}{2}m-i-1, \frac{1}{2}m-2]$ | $m+i-j-2$ | $[i+2, \frac{1}{2}m-2]$ | $m+i-j-2$ | $\frac{1}{2}m-1$ | $m-2$ |
| $\frac{1}{2}m-1$ | $\frac{1}{2}m+i$ | $\frac{1}{2}m-1$ | $\frac{1}{2}m+i$ | $[\frac{1}{2}m, i+1]$ | $m-1$ |
| $[\frac{1}{2}m, \frac{1}{2}m-i-1]$ | $m+i-j$ | $[\frac{1}{2}m, \frac{1}{2}m+i-1]$ | $m+i-j$ | $[i+2, \frac{1}{2}m+i-1]$ | $m+i-j$ |
| $\frac{1}{2}m+i$ | $\frac{1}{2}m$ | $\frac{1}{2}m+i$ | $\frac{1}{2}m$ | $\frac{1}{2}m+i$ | $\frac{1}{2}m$ |

and

$$\begin{aligned} E(H(n, m)) = E(H(n-2, m)) &\cup \{x_{-i}x_{-i+1}, x_{-i}x_{l+i-1}, x_{l+i-1}x_{l+i-2}\} \\ &\cup \{x_{-i}x_j \mid l - \tfrac{1}{2}(m+1) + i + 1 \leq j \leq l + i - 3\} \\ &\cup \{x_{l+i-1}x_j \mid -i + 1 \leq j \leq \tfrac{1}{2}(m+1) - i - 2\} \end{aligned} \tag{6}$$

and note that one of the sets is empty for $m = 5$. As in (i) it is easily verified that the r.h.s. of (3) gives the number of edges in $H(n, m)$ so that it remains only to show that $H(n, m)$ is $m$ vanishing. As before we list, without details of the inductive proof, the degrees of the vertices of $H(n, m)$.

The table for $d(x_j)$ in $H(n, m)$, $n = 2i + l$, $l = \frac{1}{2}(m-1)$, $m \equiv 1 \bmod 4$ has column-pair headings $1 \leq i \leq \frac{1}{4}(m-5)$, $\frac{1}{4}(m-1) \leq i \leq \frac{1}{2}(m-5)$ and $i \geq \frac{1}{2}(m-3)$; otherwise the table is the same as Table 3.

The table for $d(x_j)$ in $H(n, m)$, $n = 2i + l$, $l = \frac{1}{2}(m+1)$, $m \equiv 1 \bmod 4$ is the same as Table 4 except that the column-pair headings read $1 \leq i \leq \frac{1}{4}(m-5)$, $\frac{1}{4}(m-1) \leq i \leq \frac{1}{2}(m-5)$ and $i \geq \frac{1}{2}(m-3)$.

If from $H(n, m)$, $n > l$, we consecutively delete the edges $x_{-i}x_{l+i-1}$, $x_{-i}x_{-i+1}$, $x_{-i}x_j$ for $l + i - 3 \geq j \geq l - \frac{1}{2}(m+1) + i + 1$, $x_{l+i-1}x_{-i+1}$, $x_{l+i-1}x_{l+i-2}$ and $x_{l+i-1}x_j$ for $-i + 2 \leq j \leq \frac{1}{2}(m+1) - i - 2$ we are left with $H(n-2, m) + x_{-i} + x_{l+i-1}$. As outlined before we complete the proof that $H(n, m)$ is $m$-vanishing by induction in this case too and the proof of the theorem is finished.

Table 3. $d(x_j)$ in $H(n, m)$, $n = 2i + l$, $l = \frac{1}{2}(m-1)$, $m \equiv 3 \bmod 4$.

| $1 \leq i \leq \frac{1}{4}(m-3)$ | | $\frac{1}{4}(m+1) \leq i \leq \frac{1}{3}(m-5)$ | | $i \geq \frac{1}{2}(m-3)$ | |
|---|---|---|---|---|---|
| $j$ | $d(x_j)$ | $j$ | $d(x_j)$ | $j$ | $d(x_j)$ |
| $-i$ | $\frac{1}{2}(m-1)$ | $-i$ | $\frac{1}{2}(m-1)$ | $-i$ | $\frac{1}{2}(m-1)$ |
| $[-i+1, -1]$ | $\frac{1}{2}(m-1)+i+j+1$ | $[-i+1, -1]$ | $\frac{1}{2}(m-1)+i+j+1$ | $[-i+1, \frac{1}{2}(m-1)-i-2]$ | $\frac{1}{2}(m-1)+i+j+1$ |
| $0$ | $\frac{1}{2}(m-1)+i$ | $0$ | $\frac{1}{2}(m-1)+i$ | $[\frac{1}{2}(m-1)-i-1, -1]$ | $m-1$ |
| $[1, i-1]$ | $\frac{1}{2}(m-1)+i+j-1$ | $[1, \frac{1}{2}(m-1)-i-2]$ | $\frac{1}{2}(m-1)+i+j-1$ | $0$ | $m-2$ |
| $[1, \frac{1}{2}(m-1)-i-1]$ | $\frac{1}{2}(m-1)+2i-1$ | $[\frac{1}{2}(m-1)-i-1, i]$ | $m-3$ | $[1, \frac{1}{2}(m-1)-1]$ | $m-3$ |
| $[\frac{1}{2}(m-1)-i, \frac{1}{2}(m-1)-1]$ | $m+i-j-3$ | $[i+1, \frac{1}{2}(m-1)-1]$ | $m+i-j-3$ | $[\frac{1}{2}(m-1), i]$ | $m-1$ |
| $[\frac{1}{2}(m-1), \frac{1}{2}(m-1)+i-2]$ | $m+i-j-1$ | $[\frac{1}{2}(m-1), \frac{1}{2}(m-1)+i-2]$ | $m+i-j-1$ | $[i+1, \frac{1}{2}(m-1)+i-2]$ | $m+i-j-1$ |
| $\frac{1}{2}(m-1)+i-1$ | $\frac{1}{2}(m-1)+1$ | $\frac{1}{2}(m-1)+i-1$ | $\frac{1}{2}(m-1)+1$ | $\frac{1}{2}(m-1)+i-1$ | $\frac{1}{2}(m-1)+1$ |

Table 4. $d(x_j)$ in $H(n, m)$, $n = 2i + l$, $l = \frac{1}{2}(m+1)$, $m \equiv 3 \bmod 4$.

| $1 \leq i \leq \frac{1}{4}(m-7)$ | | $\frac{1}{4}(m-3) \leq \frac{1}{2}(m-5)$ | | $i \geq \frac{1}{2}(m-3)$ | |
|---|---|---|---|---|---|
| $j$ | $d(x_j)$ | $j$ | $d(x_j)$ | $j$ | $d(x_j)$ |
| $-i$ | $\frac{1}{2}(m-1)$ | $-i$ | $\frac{1}{2}(m-1)$ | $-i$ | $\frac{1}{2}(m-1)$ |
| $[-i+1, 0]$ | $\frac{1}{2}(m-1)+i+j+1$ | $[-i+1, 0]$ | $\frac{1}{2}(m-1)+i+j+1$ | $[-i+1, \frac{1}{2}(m-1)-i-2]$ | $\frac{1}{2}(m-1)+i+j+1$ |
| $[1, i]$ | $\frac{1}{2}(m-1)+i+j-1$ | $[1, \frac{1}{2}(m-1)-i-2]$ | $\frac{1}{2}(m-1)+i+j-1$ | $[\frac{1}{2}(m-1)-i-1, 0]$ | $m-1$ |
| $[i+1, \frac{1}{2}(m-1)-i-1]$ | $\frac{1}{2}(m-1)+2i$ | $[\frac{1}{2}(m-1)-i-1, i+1]$ | $m-3$ | $[1, \frac{1}{2}(m-1)-1]$ | $m-3$ |
| $[\frac{1}{2}(m-1)-i, \frac{1}{2}(m-1)-1]$ | $m+i-j-2$ | $[i+2, \frac{1}{2}(m-1)-1]$ | $m+i-j-2$ | $\frac{1}{2}(m-1)$ | $m-2$ |
| $\frac{1}{2}(m-1)$ | $\frac{1}{2}(m-1)+i$ | $\frac{1}{2}(m-1)$ | $\frac{1}{2}(m-1)+i$ | $[\frac{1}{2}(m-1)+1, i+1]$ | $m-1$ |
| $[\frac{1}{2}(m-1)+1, \frac{1}{2}(m-1)+i-1]$ | $m+i-j$ | $[\frac{1}{2}(m-1)+1, \frac{1}{2}(m-1)+i-1]$ | $m+i-j$ | $[i+2, \frac{1}{2}(m-1)+i-1]$ | $m+i-j$ |
| $\frac{1}{2}(m-1)+i$ | $\frac{1}{2}(m-1)+1$ | $\frac{1}{2}(m-1)+i$ | $\frac{1}{2}(m-1)+1$ | $\frac{1}{2}(m-1)+i$ | $\frac{1}{2}(m-1)+1$ |

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